INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Second Year, Second Semester, 2015-16
Statistics - II, Semesteral Examination, April 27, 2016

Answer any four questions **Maximum Marks: 100**

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with density $f(x|\theta) = \exp(-(x-\theta)), x > \theta$, where $-\infty < \theta < \infty$ is unknown. Consider testing at level α

$$H_0 : \theta \leq 0 \text{ versus } H_1 : \theta > 0.$$

(a) Show that the conditions required for the existence of UMP test are satisfied here.
(b) Derive the UMP test of level α.
(c) Find the minimal sufficient statistic for θ. **[25]**

2. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$ where both μ and σ^2 are unknown.
(a) Derive the generalized likelihood ratio level α test for testing $H_0 : \sigma^2 = 1$ versus $H_1 : \sigma^2 \neq 1$.
(b) Is this also the UMP level α test? Justify. **[25]**

3. Let X denote the number of independent Bernoulli(θ) trials before the first success occurs.
(a) What is the probability mass function of X?
(b) Find the Fisher Information $I_1(\theta)$ contained in X.
Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution of X with $0 < \theta < 1$ unknown.
(c) Find an estimator $T_n = T_n(X_1, \ldots, X_n)$ such that

$$\sqrt{n}\,(T_n - \theta) \longrightarrow N\left(0, \frac{1}{I_1(\theta)}\right).$$

(d) Is it true that any estimator as in (c) above is a consistent estimator of θ? Why? **[25]**

4. In an ecological study 5 independent attempts were made to photographically capture (or to camera trap) a particular tiger. The fourth attempt provided the only success. The success probability, θ, is known as the detection probability. Assume that the prior distribution on θ is Beta(0.2, 1).
(a) Derive the posterior distribution of θ given the data.
(b) Find the highest posterior density estimate of θ.
(c) Find the posterior mean and posterior standard deviation of θ.
(d) Consider testing $H_0 : \theta \leq 0.25$ versus $H_1 : \theta > 0.25$. Explain the Bayesian approach for this. **[25]**

5.(a) Let S and T be two statistics such that S has finite variance. Show that

$$\text{Var}(S) = \text{Var}(\text{E}(S|T)) + \text{E}(\text{Var}(S|T)).$$

(b) Suppose $(X_1, X_2, \ldots, X_n)$ has probability distribution P_θ, $\theta \in \Theta$. Let $\delta(X_1, X_2, \ldots, X_n)$ be an estimator of θ with finite variance. Suppose that T is sufficient for θ, and let $\delta^*(T)$, defined by $\delta^*(t) = E(\delta(X_1, X_2, \ldots, X_n)|T = t)$, be the conditional expectation of $\delta(X_1, X_2, \ldots, X_n)$ given $T = t$. Then arguing as in (a), and without applying Jensen's Inequality, prove that

$$E(\delta^*(T) - \theta)^2 \leq E(\delta(X_1, X_2, \ldots, X_n) - \theta)^2,$$

with strict inequality unless $\delta = \delta^*$ (i.e., δ is already a function of T). [25]